Forcing under the Genericity Conjecture and L

Lantze Vongkorad

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1 Introduction

The *Genericity Conjecture* is a statement from Solovay essentially asserting issues regarding forcing and genericity under $0^{\#}$. The Genericity Conjecture states:

$$\text{If } a \subseteq On \text{ and } 0^{\#} \notin L[a], \text{then } a \text{ is generic over } L. \tag{1}$$

But Jensen then refuted this conjecture, namely by showing that there exists a $a \notin L[a]$ yet a is not set-generic over L. In this paper, I will try to (Cohen) force in the presence of $0^{\#}$ and see some implications.

2 Cohen forcing and CH

We begin by setting our base model as M. The poset in Cohen forcing is:

$$\mathbb{P} := Fn(\omega_2^M \times \omega, 2) \tag{2}$$

in which "Fn" denotes finite and partial functions. ("Usual") Genericity obviously implies that a filter $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over M, in which a notion of genericity is needed to make functions in Fn "full". In particular, $\bigcup G = \kappa \times \omega \to 2$. One then shows that $M[G]$ contains κ-many sequences of functions from ω to 2 via this lemma:

Lemma 1. *If $\kappa \in M$ and G is $Fn(\kappa \times \omega, 2)$-generic over M, then $(2^\omega \geq |\kappa|)^{M[G]}$.*

And then said G can be used to show that under $M[G]$, CH is not true.

3 Genericity Conjecture

The Genericity Conjecture does not explicitly require filters, but let us do so anyway.

Theorem 2. $0^{\#} \in L[a]$ *for* $a = \omega_2$.

Proof. Let there be ω_2 extra constant symbols in the language of set theory. Interpret each $c_0, c_1, c_2, ..c_{\omega_2}$ as cardinals of the form $\aleph_0, \aleph_1, \aleph_2, ...\aleph_{\omega_2}$ via the function $f : c_n \to \aleph_n$. We also let $\hat{\sigma}_n$ denote arbitrary theorems from the theory, and we then establish an injection $g : \{\hat{\sigma} : \hat{\sigma}_n\} \to \aleph_n$. Then we can easily see that $0^{\#}$ for the set of Godel numbers of true sentences in the expanded universe. □

In addition to the fact that one cannot construct a generic set from ω_2, we had to add new ordinals, which is something that forcing cannot do.

Luckily, this conjecture was already known to be false. We get the same dilemma under the Collapse of Cardinals, Easton forcing, just to name a few.